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A uniqueness problem of the sequence product on operator effect algebra $\mathcal{E}(H)$

Liu Weihua and Wu Junde[1]

Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China

E-mail: wjd@zju.edu.cn

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Abstract

A quantum effect is an operator on a complex Hilbert space H that satisfies $\mathbf{0} \leqslant A \leqslant I$. We denote the set of all quantum effects by $\mathcal{E}(H)$. In this paper we prove theorem 4.3, the theory of the sequential product on $\mathcal{E}(H)$ which shows, in fact, that there are sequential products on $\mathcal{E}(H)$ which are not of the generalized Lüders form. This result answers Gudder's open problem negatively.

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1. Introduction

If a quantum-mechanical system $\mathcal{S}$ is represented in the usual way by a complex Hilbert space H, then a self-adjoint operator A on H such that $\mathbf{0} \leqslant A \leqslant I$ is called the *quantum effect* on H [1, 2]. Quantum effects represent yes–no measurements that may be unsharp. A set of quantum effects on H is denoted by $\mathcal{E}(H)$. The subset $\mathcal{P}(H)$ of $\mathcal{E}(H)$ consisting of orthogonal projections represents sharp yes–no measurements. Let $\mathcal{T}(H)$ be a set of trace class operators on H and $\mathcal{S}(H)$ a set of density operators, i.e. the trace class positive operators on H of unit trace, which represent the states of a quantum system. An *operation* is a positive linear mapping $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ such that for each $T \in \mathcal{S}(H)$, $0 \leqslant \mathrm{tr}[\Phi(T)] \leqslant 1$ [3–5]. Each operation Φ can define a unique quantum effect B such that for each $T \in \mathcal{T}(H)$, $\mathrm{tr}[\Phi(T)] = \mathrm{tr}[TB]$.

Let $\mathcal{B}(H)$ be a set of bounded linear operators on H; the dual mapping $\Phi^* : \mathcal{B}(H) \to \mathcal{B}(H)$ of an operation Φ is defined by the relation $\mathrm{tr}\,[T\Phi^*(A)] = \mathrm{tr}[\Phi(T)A]$, $A \in \mathcal{B}(H)$, $T \in \mathcal{T}(H)$ [4]. The effect B defined by operation Φ satisfies that $B = \Phi^*(I)$ [5].

For each $P \in \mathcal{P}(H)$ a so-called Lüders operation $\Phi_L^P : T \to PTP$ is associated, its dual is $\left(\Phi_L^P\right)^*(A) = PAP$ and the corresponding quantum effect is $\left(\Phi_L^P\right)^*(I) = P$. These operations arise in the context of ideal measurements. Moreover, each quantum effect $B \in \mathcal{E}(H)$ gives rise to a general Lüders operation $\Phi_L^B : T \to B^{\frac{1}{2}}TB^{\frac{1}{2}}$ and B is recovered as $\left(\Phi_L^B\right)^*(I) = B$ [5].

[1] Author to whom any correspondence should be addressed.

Let Φ_1, Φ_2 be two operations. The composition $\Phi_2 \circ \Phi_1$ is a new operation, called a sequential operation as it is obtained by first performing Φ_1 and then Φ_2. In general, $\Phi_2 \circ \Phi_1 \neq \Phi_1 \circ \Phi_2$. Note that for any two quantum effects $B, C \in \mathcal{E}(H)$, we have $\left(\Phi_L^C \circ \Phi_L^B\right)^*(I) = B^{\frac{1}{2}}CB^{\frac{1}{2}}$ [5, pp 26–27]. It shows that the new quantum effect $B^{\frac{1}{2}}CB^{\frac{1}{2}}$ yielded by B and C has an important physical meaning. Professor Gudder called it the sequential product of B and C, and denoted it by $B \circ C$. It represents the quantum effect produced by fist measuring A and then measuring B [6–8]. This sequential product has also been generalized to an algebraic structure called a *sequential effect algebra* [7].

Now, we introduce the abstract sequential product on $\mathcal{E}(H)$ as follows.

Let $\circ$ be a binary operation on $\mathcal{E}(H)$, i.e. $\circ$: $\mathcal{E}(H) \times \mathcal{E}(H) \to \mathcal{E}(H)$, if it satisfies the following.

(S1) The map $B \to A \circ B$ is additive for each $A \in \mathcal{E}(H)$, that is, if $B + C \leqslant I$, then $(A \circ B) + (A \circ C) \leqslant I$ and $(A \circ B) + (A \circ C) = A \circ (B + C)$.
(S2) $I \circ A = A$ for all $A \in \mathcal{E}(H)$.
(S3) If $A \circ B = \mathbf{0}$, then $A \circ B = B \circ A$.
(S4) If $A \circ B = B \circ A$, then $A \circ (I - B) = (I - B) \circ A$ and $A \circ (B \circ C) = (A \circ B) \circ C$ for all $C \in \mathcal{E}(H)$.
(S5) If $C \circ A = A \circ C, C \circ B = B \circ C$, then $C \circ (A \circ B) = (A \circ B) \circ C$ and $C \circ (A + B) = (A + B) \circ C$ whenever $A + B \leqslant I$.

If $\mathcal{E}(H)$ has a binary operation $\circ$ satisfying conditions (S1)–(S5), then $(\mathcal{E}(H), \mathbf{0}, I, \circ)$ is called a sequential operator effect algebra. Professor Gudder showed that for any two quantum effects B and C, the operation $\circ$ defined by $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$ satisfies conditions (S1)–(S5), and so is a sequential product of $\mathcal{E}(H)$, which we call the generalized Lüders form. In 2005, Professor Gudder presented 25 open problems about the general sequential effect algebras. The second problem is as follows.

Problem 1.1. [9] Is $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$ the only sequential product on $\mathcal{E}(H)$?
As we see, the five properties are based on the measurement logics and the uniqueness property has been asked many times in Gudder's paper. In this paper, we construct a new sequential product on $\mathcal{E}(H)$ which differs from the generalized Lüders form; thus, we answer the open problem negatively.

2. The sequential product on $\mathcal{E}(H)$

In this section, we study some abstract properties of the sequential product $\circ$ on $\mathcal{E}(H)$. For convenience, we introduce the following notations: if $A, B \in \mathcal{E}(H)$, we say that $A \oplus B$ is defined if and only if $A + B \leqslant I$ and define $A \oplus B = A + B$; if $A \circ B = B \circ A$, we denote $A|B$.

Lemma 2.1. *If $A, B \in \mathcal{E}(H)$, $a \in [0, 1]$, then*

$$A \circ (aB) = a(A \circ B).$$

Proof. It is clear that for $a = 1$, the conclusion is true. If $a > 0$ is a rational number, i.e. $a = \frac{m}{n}$, where n, m are positive integers, it follows from $\bigoplus_{i=1}^n \left(A \circ \frac{1}{n}B\right) = A \circ B$ that $A \circ \left(\frac{1}{n}B\right) = \frac{1}{n}(A \circ B)$; thus, $A \circ \left(\frac{m}{n}B\right) = \bigoplus_{i=1}^m A \circ \left(\frac{1}{n}B\right) = \frac{m}{n}(A \circ B)$. If $a \in [0, 1]$ is not a rational number, then for each $q = \frac{m}{n} > a$, we have $q(A \circ B) = A \circ (qB) = A \circ [(q - a)B] + A \circ (aB) \geqslant A \circ (aB)$, so $q(A \circ B) \geqslant A \circ (aB)$. Let $q \to a$; we have $a(A \circ B) \geqslant A \circ (aB)$. Similarly, we can get that $A \circ (aB) \geqslant a(A \circ B)$ by taking $q = \frac{m}{n} < a$.

So $A \circ (aB) = a(A \circ B)$. Moreover, it follows from the proof process that for $a = 0$ the conclusion is also true. □

Lemma 2.2. [9, theorem 3.4(i)] *Let $A \in \mathcal{E}(H)$ and $E \in \mathcal{P}(H)$. If $A \leqslant E$, then $A|E$ and $E \circ A = A$.*

Lemma 2.3. *If $a \in [0, 1]$, $E \in \mathcal{P}(H)$, then $aI|E$ and $(aI) \circ E = E \circ (aI) = aE$.*

Proof. Since $aE \leqslant E$, and $aE|E$ and $E \circ E = E$ by lemma 2.2, it follows from $E = E \circ I = (E \circ E) \oplus (E \circ (I - E)) = E \oplus (E \circ (I - E))$ that $E \circ (I - E) = \mathbf{0}$. Note that $E \circ (a(I - E)) \leqslant E \circ (I - E) = \mathbf{0}$, so $E \circ (a(I - E)) = \mathbf{0}$; thus, it follows from (S3) that $E|a(I - E)$. Moreover, by (S5) we have $E|a(I - E) \oplus aE = aI$, so it follows from lemmas 2.1 and 2.2 that $(aI) \circ E = E \circ (aI) = a(E \circ I) = aE$. □

Lemma 2.4. *If $E, F \in \mathcal{P}(H)$, $E \leqslant F$ and $0 \leqslant a \leqslant 1$, then $E|aF$ and $E \circ (aF) = aE$.*

Proof. It follows from $E \leqslant F$ that $I - E \geqslant I - F \geqslant a(I - F)$. By lemmas 2.2 and 2.3, we have $I - E|a(I - F)$ and $I - E|(1 - a)I$; thus, $I - E|a(I - F) \oplus (1 - a)I = I - aF$. It follows from (S4) that $E|I - aF$ and again by (S4) that $E|aF$; moreover, by lemmas 2.1 and 2.2, we have $(aF) \circ E = E \circ (aF) = a(E \circ F) = aE$. □

Lemma 2.5. *If $E \in \mathcal{P}(H)$, $A \in \mathcal{E}(H)$, $0 \leqslant a \leqslant 1$ and $A \leqslant E$, then $aE|A$ and $(aE) \circ A = A \circ (aE) = aA$.*

Proof. It follows from lemma 2.2 that $A|E$, so by (S4) we have $A|I - E$. Since $A \circ E = A = A \circ I = A \circ E \oplus A \circ (I - E)$, we have $A \circ (I - E) = \mathbf{0}$. Note that $A \circ (a(I - E)) \leqslant A \circ (I - E)$; we have $A \circ (a(I - E)) = \mathbf{0}$, so $A|a(I - E)$.

Let $\{E_\lambda\}$ be the identity resolution of A and denote

$$A_n = \sum_{i=0}^{2^n-1} \frac{i}{2^n}\left(E_{\frac{i+1}{2^n}} - E_{\frac{i}{2^n}}\right),$$

$$B_n = \sum_{i=1}^{2^n} \frac{i}{2^n}\left(E_{\frac{i}{2^n}} - E_{\frac{i-1}{2^n}}\right).$$

Note that $A \in \varepsilon(H)$, so $E_\lambda = 0$ when $\lambda < 0$ and $E_\lambda = I$ when $1 \leqslant \lambda$. Moreover, for each $n \in \mathbb{N}$, $A_n \leqslant A_{n+1}$, $B_{n+1} \leqslant B_n$, and when $n \to \infty$, $\|A_n - A\| \to 0$, $\|B_n - A\| \to 0$ [10].

Let $0 \leqslant b \leqslant 1$. Then it follows from lemmas 2.1 and 2.3 that

$$\begin{aligned}(bI) \circ A_n &= \sum_{i=1}^{2^n-1} (bI) \circ \left(\frac{i}{2^n}\right)\left(E_{\frac{i+1}{2^n}} - E_{\frac{i}{2^n}}\right) \\ &= \sum_{i=1}^{2^n-1} \left(\frac{ib}{2^n}\right)\left(E_{\frac{i+1}{2^n}} - E_{\frac{i}{2^n}}\right) = bA_n\end{aligned}$$

and

$$(bI) \circ B_n = bB_n.$$

Note that $A \geqslant A_n$, so $(bI) \circ A \geqslant (bI) \circ A_n = bA_n$. Let $n \to \infty$. Then $(bI) \circ A \geqslant bA$. Doing the same with $\{B_n\}$, we get $(bI) \circ A \leqslant bA$, so $(bI) \circ A = bA = A \circ (bI)$. That is, $A|bI$ for each $0 \leqslant b \leqslant 1$; in particular, $A|(1 - a)I$. Thus, it follows from $A|(1 - a)I + a(I - E)$ that $A|I - aE$; by (S4), we have $A|aE$. Hence, $(aE) \circ A = A \circ (aE) = a(A \circ E) = aA$. □

Lemma 2.6. *Let* $0 \leqslant a \leqslant 1$ *and* $A, B \in \mathcal{E}(H)$. *Then*

$$(aA) \circ B = A \circ (aB) = a(A \circ B).$$

Proof. It follows from lemma 2.5 that $(aA) \circ B = (A \circ (aI)) \circ B = A \circ ((aI) \circ B) = A \circ (aB) = a(A \circ B)$. □

Lemma 2.6 showed that we can write $a(A \circ B)$ for $(aA) \circ B$ and $A \circ (aB)$.

In order to obtain our main result in this section, we need to extent $\circ : \mathcal{E}(H) \times \mathcal{E}(H) \to \mathcal{E}(H)$ to $\mathcal{E}(H) \times \mathcal{S}(H) \to \mathcal{S}(H)$, where $\mathcal{S}(H)$ is the set of bounded linear self-adjoint operators on H.

Let $B \in \mathcal{E}(H)$, $A \in \mathcal{S}^{+}(H)$. Then there exists a number $M > 0$ such that $\frac{A}{M} \in \mathcal{E}(H)$. Now we define

$$B \circ A = M \left(B \circ \frac{A}{M} \right).$$

If there is another positive number M' such that $\frac{A}{M'} \in \mathcal{E}(H)$, without losing generality, we assume that $M \leqslant M'$; then $M'\big(B \circ \frac{A}{M'}\big) = M'\big(B \circ \big(\frac{M}{M'}\frac{A}{M}\big)\big) = M'\big(\frac{M}{M'}\big(B \circ \frac{A}{M}\big)\big) = M\big(B \circ \frac{A}{M}\big)$. This showed that $B \circ A$ is well defined for each bounded linear positive operator A on H.

In general, if $A \in \mathcal{S}(H)$, we can express A as $A_1 - A_2$, where A_1, A_2 are two bounded linear positive operators on H [10]. Now we define

$$B \circ A = B \circ A_1 - B \circ A_2.$$

If $A_1' - A_2'$ is another expression of A with the above properties, then $A_1 + A_2' = A_1' + A_2 = K$ is a bounded linear positive operator on H. If we take a positive real number M such that $\frac{K}{M} \in \mathcal{E}(H)$, then $B \circ (A_1 + A_2') = M\big(B \circ \big(\frac{A_1}{M} + \frac{A_2'}{M}\big)\big) = M\big(B \circ \frac{A_1}{M}\big) + M\big(B \circ \frac{A_2'}{M}\big) = B \circ A_1 + B \circ A_2'$. Similarly, $B \circ (A_1' + A_2) = B \circ A_1' + B \circ A_2$. Thus, it follows from $B \circ A_1' + B \circ A_2 = B \circ A_1 + B \circ A_2'$, $B \circ A_1 - B \circ A_2 = B \circ A_1' - B \circ A_2'$. This showed that $\circ$ is well defined on $\mathcal{E}(H) \times S(H)$.

From the above discussion, we can easily prove the following important result.

Theorem 2.7. *If* $B \in \mathcal{E}(H)$, $A_1, A_2 \in S(H)$ *and* $a \in \mathbb{R}$, *then we have*

$$B \circ (A_1 + A_2) = B \circ A_1 + B \circ A_2, B \circ (aA_1) = a(B \circ A_1).$$

3. Sequential product on $\mathcal{E}(H)$ with dim $(H) = 2$

In this section, we suppose that $\dim(H) = 2$. Now, we explore the key idea of constructing our sequential product.

Lemma 3.1. *If* $E \in \mathcal{P}(H)$, $B \in \mathcal{E}(H)$, *then* $E \circ B = EBE$.

Proof. Since E is a orthogonal projection on $\mathcal{E}(H)$ with $\dim(H) = 2$, there exists a normal basis $\{e_1, e_2\}$ of H such that $E(e_i) = \lambda_i e_i$, where $\lambda_i \in \{0, 1\}$, $i = 1, 2$. If $\lambda_i = 0, i = 1, 2$, then $E = \mathbf{0}$; if $\lambda_i = 1, i = 1, 2$, then $E = I$. It is clear that for $E = \mathbf{0}$ or $E = I$, the conclusion is true. Without losing generality, we now suppose that $\lambda_1 = 1$ and $\lambda_2 = 0$, i.e. $(E(e_1), E(e_2)) = (e_1, e_2)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Let $B \in S(H)$. Then we have $(B(e_1), B(e_2)) = (e_1, e_2)\left(\begin{smallmatrix} x & y \\ \bar{y} & z \end{smallmatrix}\right)$, where $x, z \in \mathbb{R}$ ([10]). Now we define two linear operators X and Z on H which satisfy that

$$(X(e_1), X(e_2)) = (e_1, e_2) \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(Z(e_1), Z(e_2)) = (e_1, e_2)\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}.$$

Then $X = xE$, $Z = z(I-E) \in \mathcal{E}(H)$ and it follows from (S1) and lemma 2.2 that $E \circ X = X$ and $E \circ Z = \mathbf{0}$. Denote

$$(E \circ B(e_1), E \circ B(e_2)) = (e_1, e_2)\begin{pmatrix} f(x,y,z) & g(x.y.z) \\ \overline{g(x.y.z)} & h(x,y,z) \end{pmatrix}.$$

Since $S(H)$ is a real linear space and, by theorem 2.7, that $B \to E \circ B$ is a real linear map of $S(H) \to S(H)$, f, g and h are real linear maps of vector (x, y, z) and f and g are real-valued functions of (x, y, z); thus, function $f(x, y, z)$ must have the following form [10]: $f(x, y, z) = kx + lz + n(y + \bar{y}) + \mathrm{i}m(y - \bar{y})$, where $k, l, m, n \in R$. Let $B = X$ and $B = Z$, respectively. It follows from $E \circ X = X$ and $E \circ Z = \mathbf{0}$ that $l = 0, k = 1$, so $f(x, y, z) = x + n(y + \bar{y}) + m\mathrm{i}(y - \bar{y})$. Note that when $B \in \mathcal{S}^+(H)$, $E \circ B$ should be a positive operator; hence, when $x, z \geqslant 0$ and $xz - |y|^2 \geqslant 0$, we have $f(x, y, z) \geqslant 0$. Take $y \in R$; then $f(x, y, z) = x + 2ny$. Thus, when $x, z \geqslant 0$, $y \in R$ and $xz - y^2 \geqslant 0$, $f(x, y, z) = x + 2ny \geqslant 0$. If $n \neq 0$, take $y = -\frac{1}{n}, x = 1, z = \frac{1}{n^2}$; then we have $f < 0$. This is a contradiction and so $n = 0$. Similarly, if $m \neq 0$, take $y = -\frac{i}{m}, x = 1, z = \frac{1}{m^2}$; we will get $f < 0$. This is also a contradiction and so $m = 0$. Thus, we have $f(x, y, z) = x$.

Moreover, note that $E\circ((I-E)\circ B) = (E\circ(I-E))\circ B = \mathbf{0}\circ B = \mathbf{0} = ((I-E)\circ E))\circ B = (I-E)\circ(E\circ B)$, as above; we may prove that $((I-E)\circ(E\circ B)(e_1), (I-E)\circ(E\circ B)(e_2)) = (e_1, e_2)\begin{pmatrix} 0 & 0 \\ 0 & h(x,y,z) \end{pmatrix} = (e_1, e_2)\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, $h(x, y, z) = 0$. For each $y \in \mathbb{C}$, take $x = 1, z = |y|^2$; then B is a positive operator and so $E \circ B$ is also a positive operator. Thus, we have $fh - |g|^2 \geqslant 0$. It follows from $h = 0$ that $g = 0$, so $E \circ B = X = EBE$. $\square$

Corollary 3.2. *Let $E \in \mathcal{P}(H)$, $a \in [0, 1]$ and $A = aE$. Then for each $B \in \mathcal{E}(H)$,*

$$A \circ B = (aE) \circ B = a(E \circ B) = a(EBE) = a^{\frac{1}{2}}EBa^{\frac{1}{2}}E = A^{\frac{1}{2}}BA^{\frac{1}{2}}.$$

Now, we prove the following important result.

Theorem 3.3. *Let H be a complex Hilbert space with $dim(H) = 2$, $A, B \in \mathcal{E}(H)$. If $\{e_1, e_2\}$ is a normal basis of H such that $(A(e_1), A(e_2)) = (e_1, e_2)\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$ and $(B(e_1), B(e_2)) = (e_1, e_2)\begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix}$, then there exists a $\theta \in \mathbb{R}$ such that*

$$(A \circ B(e_1), A \circ B(e_2)) = (e_1, e_2)\begin{pmatrix} a^2x & ab\,\mathrm{e}^{\mathrm{i}\theta}y \\ ab\,\mathrm{e}^{-\mathrm{i}\theta}\bar{y} & b^2z \end{pmatrix}.$$

Proof. Let $\{e_1, e_2\}$ be a normal basis of H such that $(A(e_1), A(e_2)) = (e_1, e_2)\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$ and $(B(e_1), B(e_2)) = (e_1, e_2)\begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix}$, where $0 \leqslant a, b \leqslant 1, 0 \leqslant x, 0 \leqslant z, 0 \leqslant xz - |y|^2$. Now we define a linear operator E on H such that $(E(e_1), E(e_2)) = (e_1, e_2)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; then $E \in \mathcal{P}(H)$. By corollary 3.2, we can suppose that $a, b \in (0, 1]$ and $a \neq b$. Thus, $A = a^2E + b^2(I - E)$. Denote $(A \circ B(e_1), A \circ B(e_2)) = (e_1, e_2)\begin{pmatrix} f(x,y,z) & g(x,y,z) \\ \overline{g(x,y,z)} & h(x,y,z) \end{pmatrix}$, where f, g, h are real linear functions with respect to $(x, y, z) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R}$ and f, h take values in $\mathbb{R}$. Since $E \circ (A \circ B) = (E \circ A) \circ B) = (E \circ (a^2E + b^2(I - E))) \circ B = a^2(E \circ B)$, we have

$f(x, y, z) = a^2x$. Similarly, we also have $h(x, y, z) = b^2z$. Moreover, since $E|E, E|(I-E)$, by (S5), we have $E|A$ and, by (S4), we have $(I-E)|A$; thus, $A\circ(xE) = xa^2E$, $A\circ z(I-E) = zb^2(I-E)$. This shows that g is independent of x and z, so $g(x, y, z) = \alpha y$, where $\alpha \in C$. On the other hand, if $B \in \mathcal{S}(H)$ is a positive operator, then $A \circ B$ is also a positive operator, so for each positive number x and z, and each complex number y, when $xz - |y|^2 \geqslant 0$, we have $a^2b^2xz - |\alpha y|^2 \geqslant 0$. Let $x = 1$, $z = |y|^2$. Then we get that

$$a^2b^2 - |\alpha|^2 \geqslant 0. \tag{1}$$

Let B, C be two positive operators. We show that if both $B \leqslant C$ and $C \leqslant B$ are not true, then both $A\circ B \leqslant A\circ C$ and $A\circ C \leqslant A\circ B$ are also not true. In fact, let $D = b^2E+a^2(I-E)$. Then $A|b^2E+a^2(I-E) = D$ and $A \circ D = A \circ (b^2E+a^2(I-E)) = a^2b^2I$. So if $A\circ B \leqslant A \circ C$, then $D \circ (A \circ B) \leqslant D \circ (A \circ C)$. But $D \circ (A \circ B) = (D \circ A) \circ B = a^2b^2I \circ B = a^2b^2B \leqslant D \circ (A \circ C) = a^2b^2C$; thus, we will have $B \leqslant C$. This is a contradiction. So $A \circ B \leqslant A \circ C$ is not true. Similarly, we have that $A \circ C \leqslant A \circ B$ is also not true.

Let $y \in \mathbb{C}$, $y \neq 0$, ϵ be a positive number satisfying that $a^2|y| - \epsilon > 0$. If we define $(B(e_1), B(e_2)) = (e_1, e_2)\begin{pmatrix} |y| & y \\ \bar{y} & |y| \end{pmatrix}$ and $(C(e_1, C(e_2)) = (e_1, e_2)\begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix}$, then $B, C \in \mathcal{E}(H)$, and $B \leqslant C$ and $C \leqslant B$ are both not true. Thus, we have that both $A\circ B \leqslant A\circ C$ and $A\circ B \leqslant A\circ C$ are also not true, i.e. the self-adjoint operator $A\circ B - A\circ C$ is not a positive operator. Note that $((A\circ B-A\circ C)(e_1), (A\circ B-A\circ C)(e_2)) = (e_1, e_2)\begin{pmatrix} a^2|y|-\epsilon & \alpha y \\ \overline{\alpha y} & b^2|y| \end{pmatrix}$, and $a^2|y|-\epsilon > 0$, $b^2|y| > 0$, so we have $b^2(a^2|y|-\epsilon)|y| - |\alpha y|^2 < 0$. Let $\epsilon \to 0$; we get that $|\alpha y|^2 \geqslant b^2a^2|y|^2$. Thus, we have

$$|\alpha|^2 \geqslant b^2a^2. \tag{2}$$

It follows from (1) and (2) that $|\alpha|^2 = a^2b^2$. So $|\alpha| = ab$ and $\alpha = ab\mathrm{e}^{\mathrm{i}\theta}$. □

4. A new sequential product on $\mathcal{E}(H)$

Theorem 3.2 motivated us to construct a new sequential product on $\mathcal{E}(H)$. First, we need the following.

For each $A \in \mathcal{E}(H)$, denote $R(A) = \{Ax, x \in H\}$, $N(A) = \{x, x \in H, Ax = \mathbf{0}\}$, and let P_0 and P_1 be the orthogonal projections on $\overline{R(A)}$ and $N(A)$, respectively. It follows from $A \in \mathcal{E}(H)$ that $N(A) = N(A^{1/2})$, so $\overline{R(A)} = \overline{R(A^{1/2})}$. Moreover, $P_0(H)\perp P_1(H)$ and $H = P_0(H) \oplus P_1(H)$ [10].

Denote by $f_z(u)$ the complex-valued Borel function defined on $[0, 1]$, where $f_z(u) = \exp z(\ln u)$ if $u \in (0, 1]$ and $f_z(0) = 0$. Now, we define

$$A^i = f_i(A), \quad A^{-i} = f_{-i}(A).$$

It is easy to show that $\|A^i\| \leqslant 1$, $\|A^{-i}\| \leqslant 1$ and

$$(A^i)^* = A^{-i}, \quad A^iA^{-i} = A^{-i}A^i = P_0.$$

Theorem 4.1. *Let H be a complex Hilbert space and $A, B \in \mathcal{E}(H)$. If we define $A \circ B = A^{1/2}A^iBA^{-i}A^{1/2}$, then $\circ$ satisfies conditions (S1)–(S3).*

Proof. If $A, B \in \mathcal{E}(H)$, note that $\|A^i\| \leqslant 1$ and $\|A^{-i}\| \leqslant 1$; we have

$$\|A \circ B\| = \|A^{1/2}A^iBA^{-i}A^{1/2}\| \leqslant \|A^{1/2}\|\|A^i\|\|B\|\|A^{-i}\|\|A^{1/2}\| \leqslant 1$$

and

$$< A^{1/2}A^iBA^{-i}A^{1/2}x, x >= \|B^{1/2}A^{-i}A^{1/2}x\| \geqslant 0$$

for all $x \in H$, so $A \circ B = A^{1/2}A^iBA^{-i}A^{1/2}$ is a binary operation on $\mathcal{E}(H)$. Moreover, it is clear that the map $B \to A \circ B$ is additive for each $A \in \mathcal{E}(H)$, so the operation $\circ$ satisfies (S1).

It follows from $I \circ A = I^{1/2}I^iAI^{-i}I^{1/2} = A$ that $\circ$ satisfies (S2).

If $A \circ B = A^{1/2}A^iBA^{-i}A^{1/2} = \mathbf{0}$ and we represent A and B on $H = P_0(H) \oplus P_1(H)$ by $\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, respectively, then

$$A \circ B = \begin{pmatrix} A_1^{1/2}A_1^iB_1A_1^{-i}A_1^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \mathbf{0},$$

so we have $A_1^{1/2}A_1^iB_1A_1^{-i}A_1^{1/2} = \mathbf{0}$ on $P_0(H)$, i.e. $\left(A_1^{1/2}A_1^iB_1A_1^{-i}A_1^{1/2}x, x\right) = 0$ for each $x \in P_0(H)$. Note that $R(A) = R(A^{1/2})$ and A^i is a unitary operator on $P_0(H)$, so $R(A^{1/2})$ is dense in $P_0(H)$; thus for each $y \in P_0(H)$, there is a sequence $\{z_n\} \subseteq R(A^{1/2})$ such that $z_n \to A^iy$, so there is a sequence $\{x_n\} \subseteq H$ such that $A^{1/2}x_n = z_n \to A^iy$. Let $x_n = y_n + u_n$, where $y_n \in P_0(H), u_n \in P_1(H)$. Then $A^{1/2}x_n = A^{1/2}y_n$. Thus, there is a sequence $\{y_n\}$ in $P_0(H)$ such that $A^{1/2}y_n = z_n \to A^iy$. Note that A^i is a unitary operator on $P_0(H)$, so we have $A^{-i}A^{1/2}y_n \to y$. But

$$\left\|B_1^{1/2}A_1^{-i}A_1^{1/2}y_n\right\| = \left(A_1^{1/2}A_1^iB_1A_1^{-i}A_1^{1/2}y_n, y_n\right) = 0,$$

so $B_1^{1/2}y = \mathbf{0}$ for each $y \in P_0(H)$, that is, $B_1^{1/2} = \mathbf{0}$. Since $B \in \mathcal{E}(H)$, $B_2 = \mathbf{0}$, $B_3 = \mathbf{0}$; thus, we have $B = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_4 \end{pmatrix}$, so $B \circ A = B^{1/2}B^iAB^{-i}B^{1/2} = \mathbf{0} = A \circ B$. This showed that $\circ$ satisfies (S3). □

Theorem 4.2. *Let H be a complex Hilbert space with dim $(H) < \infty, A, B \in \mathcal{E}(H)$. If we define $A \circ B = A^{1/2}A^iBA^{-i}A^{1/2}$, then $A \circ B = A^{1/2}A^iBA^{-i}A^{1/2} = B \circ A = B^{1/2}B^i AB^{-i}B^{1/2}$ if and only if $AB = BA$.*

Proof. First, it is obvious that if $AB = BA$, then $A \circ B = A^{1/2}A^iBA^{-i}A^{1/2} = B \circ A = B^{1/2}B^iAB^{-i}B^{1/2}$. Now, if $A \circ B = A^{1/2}A^iBA^{-i}A^{1/2} = B \circ A = B^{1/2}B^iAB^{-i}B^{1/2}$, we show that $AB = BA$. Note that $A \in \mathcal{E}(H)$ and dim $(H) < \infty$, so A has the form $\sum_{i=1}^n a_iE_i$, where $\sum_{k=1}^n E_k = I, a_k \geqslant 0, E_k \in \mathcal{P}(H), a_k \neq a_l, E_kE_l = \mathbf{0}$ for all $k, l = 1, 2, \ldots, n, k \neq l$. Without losing generality, we suppose that $0 \leqslant a_1 < \cdots < a_n$; then $0 \leqslant |a_1^{1/2}f_i(a_1)| < \cdots < |a_n^{1/2}f_i(a_n)|$ since $a_k^{1/2} = |a_k^{1/2}f_i(a_k)|$. It follows from the operator theory that $A^{1/2} = \sum_{k=1}^n a_k^{1/2}E_k$ and $f_i(A) = A^i = \sum_{k=1}^n f_i(a_k)E_k$, $f_{-i}(A) = A^{-i} = \sum_{k=1}^n f_{-i}(a_k)E_k$ [10]. Note that $A^{1/2}A^iBA^{-i}A^{1/2} = B^{1/2}B^iAB^{-i}B^{1/2}$, so for each $x \in H$, $(A^{1/2}A^iBA^{-i}A^{1/2}x, x) = (B^{1/2}B^iAB^{-i}B^{1/2}x, x)$; thus, we have

$$\|B^{1/2}A^{-i}A^{1/2}x\| = \|A^{1/2}B^{-i}B^{1/2}x\|. \tag{3}$$

Take $x \in E_n(H)$; then $A^{1/2}A^{-i}x = A^{-i}A^{1/2}x = a_n^{1/2}f_{-i}(a_n)x$. Note that $|a_nf_{-i}(a_n)| = |a_nf_i(a_n)| = |a_n|$, $\overline{R(B)} = \overline{R(B^{1/2})}$ and B^{-i} is a unitary operator on $\overline{R(B)}$ and $B^{-i}B^{1/2} = B^{1/2}B^{-i}$; we have

$$\begin{aligned}
\|A^{1/2}B^{1/2}B^{-i}x\|^2 &= \left\|\sum_{k=1}^n a_k^{1/2}E_kB^{1/2}B^{-i}x\right\|^2 \\
&= \sum_{k=1}^n a_k\|E_kB^{1/2}B^{-i}x\|^2 \leqslant \sum_{k=1}^n a_n\|E_kB^{1/2}B^{-i}x\|^2 \\
&= a_n\|B^{1/2}B^{-i}x\|^2 = \left\|a_n^{1/2}B^{-i}B^{1/2}x\right\|^2 \\
&= \left\|a_n^{1/2}B^{1/2}x\right\|^2 = \|B^{1/2}A^{1/2}A^{-i}x\|^2.
\end{aligned}$$

Thus, it follows from equation (3), $B^{-i}B^{1/2} = B^{1/2}B^{-i}$, $A^{-i}A^{1/2} = A^{1/2}A^{-i}$ and $0 \leqslant a_1 < \cdots < a_n$, that for each $k < n$, we have $E_k B^{1/2}B^{-i}x = \mathbf{0}$, so $B^{1/2}B^{-i}x \in E_n(H)$. Thus, we have $E_n B^{1/2}B^{-i}E_n = B^{1/2-i}E_n$. This showed that $B^{1/2}B^{-i}$ has the matrix form $\begin{pmatrix} C & D \\ \mathbf{0} & K \end{pmatrix}$ on $H = E_n(H) \oplus (I - E_n)(H)$, where $C \in \mathcal{B}(E_n(H), E_n(H)), D \in \mathcal{B}((I - E_n)(H), E_n(H)), K \in \mathcal{B}((I - E_n)(H), (I - E_n)(H))$. Note that $B \in \mathcal{E}(H)$, B has the form $\sum_{k=1}^{m} b_k F_k$ and $B^{1/2}B^{-i} = \sum_{k=1}^{m} b^{1/2} f_{-i}(b_k) F_k$, where $\sum_{k=1}^{m} F_k = I, b_k \geqslant 0, F_k \in \mathcal{P}(H), b_k \neq b_l, F_k F_l = 0$ for all $k, l = 1, 2, \ldots, m, k \neq l$. Now we define a polynomial

$$G_k(z) = \prod_{j\neq k}\left(z - b_j^{1/2} f_{-i}(b_j)\right) \Big/ \prod_{j\neq k}\left(b_k^{1/2} f_{-i}(b_j) - b_j^{1/2} f_{-i}(b_j)\right)$$

on $\mathbb{C}$. It is easy to show that for each $1 \leqslant k \leqslant m$, $G_k(B^{1/2}B^{-i}) = F_k$. Note that $B^{1/2}B^{-i}$ has an up-triangulate form, so $G_k(B^{1/2}B^{-i})$ has also an up-triangulate form. But F_k is a self-adjoint operator, so F_k has a diagonal matrix form on $E_n(H) \oplus (I - E_n)(H)$. This implies that F_k commutes with E_n for each k, so B commutes with E_n. Denote $A_0 = A - a_n E_n$; we then still have $A_0 \circ B = B \circ A_0$ as discussed before. Thus, we get that B commutes with E_{n-1}. Continuously, we will have that B commutes with all E_k and so with A. In this case, we have $A \circ B = AB$. □

Our main result is as follows.

Theorem 4.3. *Let H be a complex Hilbert space with $dim(H) < \infty$ and $A, B \in \mathcal{E}(H)$. If we define $A \circ B = A^{1/2}A^{i}BA^{-i}A^{1/2}$, then $\circ$ is a sequential product on $\mathcal{E}(H)$.*

Proof. By theorem 4.1, we only need to prove that $\circ$ satisfies (S4) and (S5). In fact, if $A|B$, i.e. $A \circ B = A^{1/2}A^{i}BA^{-i}A^{1/2} = B \circ A = B^{1/2}B^{i}AB^{-i}B^{1/2}$, then it follows from theorem 4.2 that A commutes with B and of course $I - B$, so $A|I - B$. If $C \in \mathcal{E}(H)$, we have

$$\begin{aligned} A \circ (B \circ C) &= A^{\frac{1}{2}}A^{i}B^{\frac{1}{2}}B^{i}CB^{-i}B^{\frac{1}{2}}A^{-i}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}B^{\frac{1}{2}}A^{i}B^{i}CA^{-i}B^{-i}A^{\frac{1}{2}}B^{\frac{1}{2}} \\ &= (AB)^{\frac{1}{2}}(AB)^{i}C(AB)^{-i}(AB)^{\frac{1}{2}} \\ &= (AB) \circ C = (A \circ B) \circ C. \end{aligned}$$

So (S4) is satisfied. □

Moreover, if $C|B$ and $C|A$, then $C(AB) = ACB = (AB)C$, $C(A \oplus B) = (B + A)C$, so it is easy to prove that $C(A \circ B) = (A \circ B)C$; thus, by theorem 4.2, we have $C|A \circ B$ and $C|(A \oplus B)$ whenever $A \oplus B$ is defined. This showed that (S5) hold.

By using theorem 4.3, we can prove the following corollary.

Corollary 4.4. *Let H be a complex Hilbert space with $dim(H) = 2$, $A, B \in \mathcal{E}(H)$. Take a normal basis $\{e_1, e_2\}$ of H such that $(A(e_1), A(e_2)) = (e_1, e_2)\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$ and $(B(e_1), B(e_2)) = (e_1, e_2)\begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix}$. When $a, b > 0$, define*

$$((A \circ B)(e_1), (A \circ B)(e_2)) = (e_1, e_2)\begin{pmatrix} a^2 x & ab\,\mathrm{e}^{\mathrm{i}\theta} y \\ ab\,\mathrm{e}^{-\mathrm{i}\theta}\bar{y} & b^2 z \end{pmatrix},$$

where $\theta = \ln a^2 - \ln b^2$; when $a > 0, b = 0$, define

$$((A \circ B)(e_1), (A \circ B)(e_2)) = (e_1, e_2)\begin{pmatrix} a^2 x & 0 \\ 0 & 0 \end{pmatrix};$$

when $a = 0, b > 0$*, define*

$$((A \circ B)(e_1), (A \circ B)(e_2)) = (e_1, e_2)\begin{pmatrix} 0 & 0 \\ 0 & b^2 z \end{pmatrix};$$

thus, $\circ$ *is a sequential product of* $\mathcal{E}(H)$.

Remark 1. In conclusion, we construct a new sequential product $A \circ B = A^{\frac{1}{2}} A^i B A^{-i} A^{\frac{1}{2}}$ on $\varepsilon(H)$ with dim $(H) < \infty$, which is different from the generalized Lüders form $A^{\frac{1}{2}} B A^{\frac{1}{2}}$. In this proof, we can also get a more general one $A \circ B = A^{\frac{1}{2}} A^{ti} B A^{-ti} A^{\frac{1}{2}}$ for $t \in R$. It indicates that with the measurement rule (S1)–(S5), there can be a time parameter t to describe the phase change. In particular, if $\dim(H) = 2$, $A \in \mathcal{E}(H)$ and $\{e_1, e_2\}$ is a normal basis of H such that $(A(e_1), A(e_2)) = (e_1, e_2)\left(\begin{smallmatrix} a^2 & 0 \\ 0 & b^2 \end{smallmatrix}\right)$, then when $a > 0, b > 0$ and $a \neq b$, corollary 4.4 shows that $\theta = (\ln a^2 - \ln b^2)t$ can be used to describe the phase-changed phenomena of the quantum effect $A \circ B$. As the proof shows, it is the only form that the sequential product can be of. This is much more important in physics.

Remark 2. As we know, in the quantum computation and quantum information theory, if $(A_i)_{i\in\mathbb{N}}$ is a sequence of bounded linear operators on H satisfying $\sum_{i=1}^{n} A_i A_i^* = I$, then the operators $A_i, i \in \mathbb{N}$, are called the operational elements of the quantum operation $U : \mathcal{T}(H) \to \mathcal{T}(H)$ defined by

$$U(\rho) = \sum_{n}^{n} A_i \rho A_i^*,$$

where $\mathcal{T}(H)$ is the set of trace class operators. Any trace preserving, normal, completely positive map has the above form. This is very important in describing dynamics, measurements, quantum channels, quantum interactions, quantum error, correcting codes, etc [12]. If $(A_i)_{i\in\mathbb{N}}$ is a set of quantum effects with $\sum_{i=1}^{n} A_i = I$, then the transformation $U'(\rho) = \sum_{j=1}^{n} A_j^{\frac{1}{2}} A_j^{ti} \rho A_j^{-ti} A_j^{\frac{1}{2}}$ is a well-defined quantum operation since $\sum_{j=1}^{n} A_j^{\frac{1}{2}} A_j^{ti} A_j^{-ti} A_j^{\frac{1}{2}} = \sum_{i=1}^{n} A_i = I$. So this new sequential product yields a natural and interesting quantum operation.

Remark 3. Theorem 4.3 indicates that conditions (S1)–(S5) of the sequential product of $\mathcal{E}(H)$ are not sufficient to characterize the generalized Lüders form $A^{\frac{1}{2}} B A^{\frac{1}{2}}$ of A and B. Recently, Professor Gudder presented a characterization of the sequential product of $\mathcal{E}(H)$ is the generalized Lüders form [11].

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